复 变 函 数

课后答案

第一章

1 (1) 解 $x+1+i(y-3)=(1+i)(5+3i)$

$$x+1+i(y-3)=2+8i$$

$$\begin{cases} x+1=2 \\ y-3=8 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=11 \end{cases}$$

(2) 解 $(x+y)^2 i+6i-x=-y+5(x+y)i-1$

$$\begin{cases} (x+y)^2+6=5(x+y) \\ -x=-y-1 \end{cases}$$

解之得

$$\begin{cases} x=\frac{3}{2} \\ y=\frac{1}{2} \end{cases} \text{或} \begin{cases} x=2 \\ y=1 \end{cases}$$

2. (1) i^8+i-4i^{21}

$=(i^2)^4+i-4[(i^2)^{10}]\cdot i$

$=1+i-4i$

$=1-3i$

(2) $i^{100}+2\cdot i^{-9}-3i^{-15}$

$=(i^2)^{50}+2\frac{1}{(i^2)^4\cdot i}-3\cdot\frac{1}{(i^2)^7\cdot i}$

$=1-2i-3i$

$=1-5i$

3. (1) $z=\frac{i^3}{1-i}+\frac{1-i}{i}$

$$=\frac{i^3(1+i)}{(1-i)(1+i)}+\frac{(1-i)\cdot i}{i\cdot i}$$

$$=\frac{-i+1}{2}+(-1-i)$$

$$=-\frac{1}{2}-\frac{3}{2}i$$

$|z|=\frac{\sqrt{10}}{2}$ $\arg z=\arctan 3-\pi$.

(2). $$\frac{(3+4i)(2-5i)}{2i}$$

$$= \frac{6+8i-15i-20i^2}{2i}$$

$$= \frac{26-7i}{2i}$$

$$= -\frac{7}{2} - 13i$$

$$|z| = \sqrt{\left(\frac{7}{2}\right)^2 + 13^2} = \frac{5\sqrt{29}}{2}$$

$$\arg z = \arctan\frac{26}{7} - \pi$$

(3). $$z = \left(\frac{3-4i}{1+2i}\right)^2$$

$$= \left(\frac{(3-4i)(1-2i)}{(1+2i)(1-2i)}\right)^2$$

$$= \left(\frac{-5-10i}{5}\right)^2$$

$$= (1+2i)^2 = -3+4i$$

$$|z| = \sqrt{3^2+4^2} = 5$$

$$\arg z = \arctan\frac{4}{-3} + \pi = -\arctan\frac{4}{3} + \pi$$

(4) $z = \dfrac{i}{(i-1)(i-2)(i-3)}$

$$\frac{1}{z} = \frac{(i-1)(i-2)(i-3)}{i}$$

$$= \frac{(i^2-3i+2)(i-3)}{i}$$

$$= \frac{(1-3i)(i-3)}{i}$$

$$= \frac{1-3i^2-3+9i}{i} = 10$$

$\therefore z = \dfrac{1}{10}$ $|z| = \dfrac{1}{10}$ $\arg z = 0$

4. 证明：

(1) $\bar{\bar{z}} = z$

证明：设 $z = x+iy$ $\bar{z} = x-iy$ $\bar{\bar{z}} = x+iy = z$

由于 x，y 可以任意取值，所以得证.

(2) $|z|^2 = z \cdot \bar{z}$

证明：设 $z = x+iy$. $\bar{z} = x-iy$.

$z \cdot \bar{z} = (x+iy)(x-iy) = x^2+y^2 = (\sqrt{x^2+y^2})^2 = |z|^2$

由于 x，y 可以取任意值，所以得证.

(3) $Re(z) = \dfrac{z+\bar{z}}{2}$

证明：设 $z = x+iy$.

$$\frac{z+\bar{z}}{2}=\frac{x+iy+x-iy}{2}=x=\mathrm{Re}(z).$$

由于 x，y 可以取任意值，所以得证。

(4) $\mathrm{Im}(z)=\dfrac{z-\bar{z}}{2i}$

证明：设 $z=x+iy$，

则 $$\frac{z-\bar{z}}{2i}=\frac{x+iy-(x-iy)}{2i}=y=\mathrm{Im}(z)$$

由于 x，y 可以取任意值，所以得证

5. 不成立，例如 $z=i$ 时。当 z 为实数时，(即虚部为零时)等式成立。

6. 设 $z=x+iy$

则 $$\frac{z-1}{z+1}=\frac{(x-1)+iy}{(x+1)+iy}=\frac{[(x-1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]}$$

$$=\frac{x^2+y^2-1}{(x+1)^2+y^2}+i\frac{2y}{(x+1)^2+y^2}$$

$\therefore$ 实部 $\dfrac{x^2+y^2-1}{(x+1)^2+y^2}$ 虚部 $\dfrac{2y}{(x+1)^2+y^2}$

7. (1) $5i$

$$5i=5\left(\cos\frac{\pi}{2}+i\cdot\sin\frac{\pi}{2}\right)=5\cdot e^{\frac{\pi}{2}i}$$

(2) $1+\sqrt{3}i$

$$1+\sqrt{3}i=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2\left(\cos\frac{\pi}{3}+i\cdot\sin\frac{\pi}{3}\right)=2\cdot e^{i\frac{\pi}{3}}$$

(3) $-2 = 2(-1) = 2(\cos\pi + i\cdot\sin\pi) = 2\cdot e^{i\cdot\pi}$

(4) $\sqrt{3} - i = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2\left(\cos\left(\frac{\pi}{6}\right) + i\cdot\sin\left(-\frac{\pi}{6}\right)\right) = 2\cdot e^{-\frac{\pi}{6}i}$

(5) $-2+5i =$

(6) $-2-i =$

8. (1) $3i(\sqrt{3}-i)(1+\sqrt{3}i)$

$= 3i(\sqrt{3} - i + 3i + \sqrt{3})$

$= -6 + 6\sqrt{3}i$

(2) $\frac{2i}{i-1}$

$= \frac{2i(i+1)}{(i-1)(i+1)}$

$= \frac{2i(i+1)}{-2}$

$= 1-i$

(3) $\frac{3}{(\sqrt{3}-i)^2}$

$= \frac{3}{4\cdot e^{-i\cdot\frac{\pi}{3}}}$

$= \frac{3e^{i\cdot\frac{\pi}{3}}}{4}$

$= \frac{3}{8} + \frac{3\sqrt{3}}{8}i$

(4) $(2-2i)^{\frac{1}{3}}$

$= \left(2^{\frac{3}{2}}\cdot e^{-\frac{\pi}{4}i}\right)^{\frac{1}{3}}$

$= 2^{\frac{1}{2}}\cdot e^{-\frac{\pi}{12}i}$

(7) $\sqrt[6]{-1} = (e^{i\cdot\pi})^{\frac{1}{6}} = e^{i\cdot\frac{\pi}{6}}$

(8) $(i-\sqrt{3})^{\frac{1}{5}}$

$= \left(2\cdot e^{i\cdot\frac{5}{6}\pi}\right)^{\frac{1}{5}} = 2^{\frac{1}{5}}\cdot e^{i\cdot\frac{\pi}{6}}$

(5) $z=\frac{1+\sqrt{3}i}{2}=e^{i\cdot\frac{\pi}{3}}$

$z^2=\left(e^{i\cdot\frac{\pi}{3}}\right)^2=e^{i\cdot\frac{2}{3}\pi}$

$z^4=\left(e^{i\cdot\frac{\pi}{3}}\right)^4=e^{i\cdot\frac{4}{3}\pi}$

(6) $\frac{(\cos5\varphi+i\cdot\sin5\varphi)^2}{(\cos3\varphi-i\cdot\sin3\varphi)^3}$

$=\frac{(e^{i\cdot5\varphi})^2}{(e^{i\cdot(-3\varphi)})^2}=e^{i\cdot16\varphi}$

9. 证明：左边 $=|z_1+z_2|^2+|z_1-z_2|^2$

$=(z_1+z_2)\overline{(z_1+z_2)}+(z_1-z_2)\overline{(z_1-z_2)}$

$=(z_1+z_2)(\overline{z_1}+\overline{z_2})+(z_1-z_2)(\overline{z_1}-\overline{z_2})$

$=2(z_1\cdot\overline{z_1}+z_2\cdot\overline{z_2})=2(|z_1|^2+|z_2|^2)=$ 右边

∴ 得证.

10. 证：$|z|=\sqrt{x^2+y^2}$ $|z|^2=x^2+y^2=|x|^2+|y|^2$

先证明 $\frac{|x|+|y|}{\sqrt{2}}\le|z|$

即证 $[|x|+|y|]^2\le2(|x|^2+|y|^2)$

即证 $2|x|\cdot|y|\le|x|^2+|y|^2$

即证 $(|x|-|y|)^2\geqslant0$

而此式显然成立.

∴ $\frac{|x|+|y|}{\sqrt{2}}\le|z|$ 得证

再证右半边式子 $|z| \le |x|+|y|$

即证 $(|z|)^2 \le (|x|+|y|)^2$

即证 $|x|^2+|y|^2 \le |x|^2+|y|^2+2|x|\cdot|y|$

即证 $2|x|\cdot|y| \geq 0$

而此式显然成立 ∴得证 $|z| \le |x|+|y|$

∴综上所述，$\frac{|x|+|y|}{2} \le |z| \le |x|+|y|$.

11. $\frac{z_2-z_1}{z_3-z_1} = \frac{z_1-z_3}{z_2-z_3}$

则 $\left|\frac{z_2-z_1}{z_3-z_1}\right| = \left|\frac{z_1-z_3}{z_2-z_3}\right|$

则 $\frac{|z_2-z_1|}{|z_3-z_1|} = \frac{|z_1-z_3|}{|z_2-z_3|}$

$|z_3-z_1|^2 = |z_2-z_1|\cdot|z_2-z_3|$ ①

把原式变形

$$\frac{z_2-z_3+z_3-z_1}{z_3-z_1} = \frac{z_1-z_2+z_2-z_3}{z_2-z_3}$$

$$\frac{z_2-z_3}{z_3-z_1} = \frac{z_1-z_2}{z_2-z_3}$$

$$\frac{|z_2-z_3|}{|z_3-z_1|} = \frac{|z_1-z_2|}{|z_2-z_3|}$$

$|z_2-z_3|^2 = |z_1-z_2|\cdot|z_3-z_1|$ ②

用①除以②式，可得

$$\frac{|z_3-z_1|^2}{|z_2-z_3|^2}=\frac{|z_2-z_1||z_2-z_3|}{|z_1-z_2|\cdot|z_3-z_1|}，可得$$

$$|z_3-z_1|^3=|z_2-z_3|^3$$

可得 $|z_3-z_1|=|z_2-z_3|$

同理可证 $|z_3-z_1|=|z_1-z_2|$

∴可证 $|z_2-z_1|=|z_3-z_1|=|z_2-z_3|$.

12. (1) 解：$z^n+\frac{1}{z^n}=e^{i\cdot n\theta}+e^{i\cdot(-n\theta)}$

$=\cos n\theta+i\cdot\sin n\theta+\cos(-n\theta)+i\cdot\sin(-n\theta)$

$=2\cos n\theta.$

(2) 证明：$z^n-\frac{1}{z^n}=e^{i\cdot n\theta}-e^{i\cdot(-n\theta)}$

$=\cos n\theta+i\cdot\sin n\theta-(\cos(-n\theta)+i\cdot\sin(-n\theta))$

$=2i\cdot\sin n\theta$

13. $z^4+a^4=0$ （$a>0$，为实数）.

$z^4=-a^4=a^4(\cos\pi+i\cdot\sin\pi)$

$z=a\cdot(\cos\frac{\pi+2k\pi}{4}+i\cdot\sin\frac{\pi+2k\pi}{4})$.

$(k=0,1,2,3)$.

$z_1=a(\cos\frac{\pi}{4}+i\cdot\sin\frac{\pi}{4})$ $z_2=a\cdot(\cos\frac{3}{4}\pi+i\cdot\sin\frac{3}{4}\pi)$

$z_3=a\cdot(\cos\frac{5}{4}\pi+i\cdot\sin\frac{5}{4}\pi)$ $z_4=a\cdot(\cos\frac{7}{4}\pi+i\cdot\sin\frac{7}{4}\pi)$.

14. $\frac{1}{2}(\sqrt{2}+i\sqrt{2})=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i=1\times(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

$$\sqrt[3]{\frac{1}{2}(\sqrt{2}+i\sqrt{2})}=1\times(\cos\frac{\frac{\pi}{4}+2k\pi}{3}+i\sin\frac{\frac{\pi}{4}+2k\pi}{3})$$

$(k=0,1,2)$

$k=0$, $\sqrt[3]{\frac{1}{2}(\sqrt{2}+i\sqrt{2})}=\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}$

$k=1$, $\sqrt[3]{\frac{1}{2}(\sqrt{2}+i\sqrt{2})}=\cos\frac{9\pi}{12}+i\sin\frac{9\pi}{12}$

$k=2$, $\sqrt[3]{\frac{1}{2}(\sqrt{2}+i\sqrt{2})}=\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}$

15. (1) √ (2) × (3) √ (4) × (5) ×

(6) × (7) × (8) × (9) × (10) ×

16. (1) 以 $-i$ 为中心，半径为2的圆周.

(2) 以 $2i$ 为中心，半径为1的圆周及其外部区域

(3) 以原点为中心，半径为 $\frac{1}{3}$ 的圆的外部区域，不含边界.

(4) 直线 $y=3$;

(5) 直线 $y=-1$;

(6) 直线 $y=-x$;

(7) 直线 $x=2$ 及其左侧平面

(8) 右半平面（不包括 y 轴）

(9) 以 -3 和 -1 为焦点，长轴为4的椭圆；

(10) 以 i 为起点的射线，$y=x+1\ (x>0)$.

17 (1) 不包含实轴的下半平面，是无界、开的单连通区域。

(2) 抛物线 $y^2=1-2x$ 为边界的左侧内部区域（不含边界），是无界、开的、单连通域。

(3) 由射线 $\theta=1$，$\theta=1+\pi$ 构成的角形线，即一半平面（不包括两射线在内），是无界、开的单连通域；

(4) 中心在 $z=-\frac{17}{15}$，半径为 $\frac{8}{15}$ 的圆周的外部区域（不含边界），是无界、开的、多连通域；

(5) 以原点为中心，1和3分别为内、外半径的圆环所围区域内部，不含小圆边界，包含大圆边界，是有界、半开半闭的多连通域。

(6) 以 i 为中心，1和2分别为内外半径的圆环所围区域内部，包含边界，是有界、闭的、多连通域。

(7) 双曲线 $4x^2-\frac{4}{15}y^2=1$ 的左边分支的左侧区域，（不含边界），是无界、开的单连通域；

(8) 圆 $(x-2)^2+(y+1)^2=9$ 及其内部区域，是有界、闭的单连通域；

(9) 椭圆 $\frac{x^2}{9}+\frac{y^2}{5}=1$ 及其内部区域，是有界、闭的单连通域。

(10) $1<x<2$ 的带形区域，是无界，开的单连通域.

18. 解：设 $a=u+vi$，$z=x+iy$，由于 a 为非零复常数，

$\therefore$ u, v 不同时为 0.

把 a, z 代入题中等式，可得

$(u+vi)(x-iy)+(u-vi)(x+iy)=c$

整理，得

$2ux+2vy=c$，

由于 u, v 不同时为零，所以 z 平面上的方程

可以写成 $a\bar{z}+\bar{a}z=c$.

19. 解：$\begin{cases} x+iy=z \\ x-iy=\bar{z} \end{cases} \Rightarrow \begin{cases} x=\dfrac{z+\bar{z}}{2} \\ y=\dfrac{z-\bar{z}}{2i} \end{cases}$

将 x, y 代入等式，整理得

$$a\left(\left(\frac{z+\bar{z}}{2}\right)^2+\left(\frac{z-\bar{z}}{2i}\right)^2\right)+b\cdot\frac{z+\bar{z}}{2}+c\cdot\frac{z-\bar{z}}{2i}+d=0$$

即 $a\cdot z\cdot\bar{z}+\left(\frac{b}{2}+\frac{c}{2i}\right)z+\left(\frac{b}{2}-\frac{c}{2i}\right)\bar{z}+d=0$.

20 解：(1) $w_1=i^3=-i$

$w_2=(1-i)^3=-2-2i$

$w_3=(\sqrt{3}+i)^3=8i$；

(2) $0<\arg w<\pi$.

21 (1) $z = t + 2ti$

$x = t, \quad y = 2t.$

$y = 2x$

(2) $x = a\cdot\cos t, \quad y = b\cdot\sin t$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

(3) $x = t, \quad y = \frac{1}{t}$

$xy = 1$

(4). $z = a\cdot(\cos t + i\cdot\sin t) + b(\cos t - i\cdot\sin t)$

$z = a\cdot\cos t + b\cdot\cos t + (a\cdot\sin t - b\cdot\sin t)i$

$x = (a + b)\cos t, \quad y = (a-b)\sin t$

$\frac{x^2}{(a+b)^2} + \frac{y^2}{(a-b)^2} = 1;$

22. (1) $z(t) = 2\cos t + i 2\sin t \qquad 0 \le t \le 2\pi$

(2) $z(t) = 3\cos t + 1 + i\cdot 3\cdot\sin t \qquad 0 \le t \le 2\pi$

(3) $z(t) = t + 4i \qquad -\infty < t < +\infty$

(4) $z(t) = 2 + ti \qquad -\infty < t < +\infty$

(5) $z(t) = t + ti \qquad -\infty < t < +\infty$

23 解：设 $w=u+iv$

(1) $z=2\cos t+i2\sin t$

$$w=\frac{1}{z}=\frac{\cos t}{2}-\frac{\sin t}{2}i$$

$\therefore$ $u^2+v^2=\frac{1}{4}$ 圆周

(2). $z=x+iy$

$$w=\frac{1}{z}=\frac{x}{x^2+y^2}-\frac{y}{x^2+y^2}i$$

$u+v=0$ 为直线

(3) $z=1+iy$

$$w=\frac{1}{z}=\frac{1-iy}{1+y^2}$$

$$u=\frac{1}{1+y^2}\qquad v=\frac{-y}{1+y^2}$$

$$\left(u-\frac{1}{2}\right)^2+(v)^2=\frac{1}{4}$$

(4) $z=x+3i$

$$w=\frac{1}{z}=\frac{1}{x+3i}=\frac{x-3i}{x^2+9}$$

$$u=\frac{x}{x^2+9}\qquad v=\frac{-3}{x^2+9}$$

$$\left(\frac{-3}{x^2+9}+\frac{1}{6}\right)^2+\left(\frac{x}{x^2+9}\right)^2=\frac{1}{36}$$

$$\left(v+\frac{1}{6}\right)^2+u^2=\frac{1}{36}$$

(5) $x=1+\cos t$，$y=\sin t$

$$w=\frac{1}{z}=\frac{1}{2}-\frac{\sin t}{2(1+\cos t)}i$$

直线：$u=\frac{1}{2}$

24 证明：设 $z=x+iy$，代入 $f(z)$ 整理，

$$f(z)=\frac{1}{2i}\left(\frac{x+iy}{x-iy}-\frac{x-iy}{x+iy}\right)$$

$$=\frac{2xy}{x^2+y^2}$$

$$\lim_{\substack{x\to 0\\ y=kx}} f(x)=\lim_{x\to 0}\frac{2x\cdot kx}{x^2+k^2x^2}=\frac{2k}{1+k^2}$$

随 k 取值的不同，$\frac{2k}{1+k^2}$ 的取值不同，∴ f(z) 在原点处无极限

25. 当 $x<0$，$y>0$ 时 $\lim\limits_{y\to 0^+}\arg z=\pi$

当 $x<0$，$y<0$ 时 $\lim\limits_{y\to 0^-}\arg z=-\pi$

$$\lim_{y\to 0^+}\arg z\neq\lim_{y\to 0^-}\arg z$$

∴ $f(z)=\arg z$ 在原点与负实轴上不连续

26. 解：设 $f(z)=u+iv$

∵ f(z) 在 z_0 处连续　u，v 在 z_0 处 连续

$\overline{f(z)}=u-iv$，　∴ u，$-v$ 也在 z_0 处连续

∴ $\overline{f(z)}$ 在 z_0 处连续

$|f(z)|=\sqrt{u^2+v^2}$，　此为关于 u，v 的多项式，∵ u，v 连续

∴ $|f(z)|=\sqrt{u^2+v^2}$ 在 z_0 处也连续.

27 证明：由于 $f(z)$ 在 z_0 处连续

$\therefore \exists \delta$，当 $|x-x_0|<\delta$ 时，

$\lim\limits_{x\to x_0} f(z) = f(z_0) \neq 0$

由此得证 f 在 z_0 的某邻域使该领域内 $f(z)\neq 0$.

28. (1) $\lim\limits_{z\to 2+i} \dfrac{\bar{z}}{z}$

$= \dfrac{2-i}{2+i} = \dfrac{3-4i}{5}$

(2) 此题有误，无极限.

第二章

1. (1) $u=x^2$，$v=-y$

$\dfrac{\partial u}{\partial x}=2x$ $\dfrac{\partial v}{\partial y}=-1$ $\dfrac{\partial u}{\partial y}=0$，$\dfrac{\partial v}{\partial x}=0$.

$\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y} \Rightarrow 2x=-1$ $x=-\dfrac{1}{2}$

在直线 $x=-\frac{1}{2}$ 上可导，但在复平面上处处不解析.

(2) $u=2x^3$，$v=3y^3$

$\dfrac{\partial u}{\partial x}=6x^2$ $\dfrac{\partial v}{\partial y}=9y^2$

$\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y} \Rightarrow$ $\sqrt{2}x\pm\sqrt{3}y=0$ 上可导，但在复平面上

处处不解析.

(3) $u=xy^2$, $v=x^2y$.

$\frac{\partial u}{\partial x}=y^2$, $\frac{\partial v}{\partial y}=x^2$ $\frac{\partial u}{\partial y}=2xy$. $\frac{\partial v}{\partial x}=2xy$.

$y^2=x^2$, $2xy=-2xy \Rightarrow x=0, y=0$.

∴在$z=0$处可导，但在复平面上处处不解析.

(4) $u=x^3-3xy^2$, $v=3x^2y-y^3$

$\frac{\partial u}{\partial x}=3x^2-3y^2$, $\frac{\partial v}{\partial y}=3x^2-3y^2$

$\frac{\partial u}{\partial y}=-6xy$. $\frac{\partial v}{\partial x}=6xy$

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$. $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ (性质在复平面上).

在复平面上处处可导，处处解析.

2. $u=my^3+nx^2y$ $v=x^3+lxy^2$

$\frac{\partial u}{\partial x}=2nxy$ $\frac{\partial v}{\partial y}=2lxy$

$\frac{\partial u}{\partial y}=3my^2+nx^2$, $\frac{\partial v}{\partial x}=3x^2+ly^2$

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ $\Rightarrow$

$$\begin{cases} 2n=2l \\ 3m=-l \\ n=-3 \end{cases} \Rightarrow \begin{cases} n=-3 \\ l=-3 \\ m=1 \end{cases}$$

3(1) 解：$z=0$ 或 $z^2=-1$　(2) $z+1=0$ 或 $z^2+1=0$

$z=-1$ 或 $z=\pm i$

∴ $z=0$ 或 $z=\pm i$

4. 解：(1). $f'(z)=\frac{\partial u}{\partial x}+i\cdot\frac{\partial v}{\partial x}$

$|f(z)|=\sqrt{u^2+v^2}$

$$\frac{\partial|f(z)|}{\partial x}=\frac{1}{\sqrt{u^2+v^2}}\left(u\cdot\frac{\partial u}{\partial x}+v\cdot\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial|f(z)|}{\partial y}=\frac{1}{\sqrt{u^2+v^2}}\left(u\cdot\frac{\partial u}{\partial y}+v\cdot\frac{\partial v}{\partial y}\right)$$

左边 $=\frac{1}{u^2+v^2}\left[\left(u\cdot\frac{\partial u}{\partial x}+v\cdot\frac{\partial v}{\partial x}\right)^2+\left(u\frac{\partial u}{\partial y}+v\cdot\frac{\partial v}{\partial y}\right)^2\right]$

右边 $=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$

∵ $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ 代入左边，

左边 $=\left[u^2\cdot\left(\frac{\partial u}{\partial x}\right)^2+v^2\left(\frac{\partial u}{\partial x}\right)^2+2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\right.$

$\left.u^2\left(-\frac{\partial v}{\partial x}\right)^2+v^2\left(\frac{\partial u}{\partial x}\right)^2+2uv\frac{\partial u}{\partial x}\left(-\frac{\partial v}{\partial x}\right)\right]\cdot\frac{1}{u^2+v^2}$

$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=$ 右边

∴ 得证

(2) $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=\frac{\partial(-u)}{\partial y}$，$\frac{\partial(-u)}{\partial x}=-\frac{\partial v}{\partial y}$

∴ $-u$ 为 v 的共轭调和函数

(此处应在第二章有介绍，调和函数)。

(3). $\frac{\partial^2 |f(z)|^2}{\partial x^2} = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)^2 + 2u\cdot\frac{\partial^2 u}{\partial x^2} + 2v\cdot\frac{\partial^2 v}{\partial x^2}$

$\frac{\partial^2 |f(z)|^2}{\partial y^2} = 2\left(\frac{\partial u}{\partial y}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2} + 2v\frac{\partial^2 v}{\partial y^2}$

由于在区域内解析, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ $\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{-\partial x \partial y}$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ 同理可得 $\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0$

∴ 左边 $= 4(u_x^2 + v_x^2) = 4|f'(z)|^2 =$ 右边

5 设 $f(z) = u + iv$

$\overline{if(z)} = iu - v = -v + iu$

∵ $f(z)$ 解析

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial (-v)}{\partial x} = -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, $\frac{\partial(-v)}{\partial y} = -\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

∴ 可证 $\overline{if(z)}$ 在D内也解析.

6. (1). $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(6xy + 3x^2 - 3y^2)$

$v = \int \frac{\partial v}{\partial x} dx = \int (-6xy + 3x^2 - 3y^2) dx = (-3x^2 y + x^3 - 3xy^2 + \varphi(y))$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow$ $3x^2 + 6xy - 3y^2 = -3x^2 + 6xy - \varphi'(y)$

$\varphi'(y) = 3y^2$

$\varphi(y) = \int \varphi'(y) dy = y^3 + C$

$v = 3x^2 y - x^3 + 3xy^2 - y^3 - C$

27 证明：由于 $f(z)$ 在 z_0 处连续

$\therefore \exists \delta$，当 $|x-x_0|<\delta$ 时，

$$\lim_{x\to x_0} f(z) = f(z_0) \neq 0$$

由此得证 $f(z)$ 在 z_0 的某邻域使该邻域内 $f(z)\neq 0$.

28. (1) $\lim\limits_{z\to 2+i} \dfrac{\bar z}{z}$

$$= \frac{2-i}{2+i} = \frac{3-4i}{5}$$

(2) 此题有误，无极限

第二章

1. (1) $u=x^2$，$v=-y$

$$\frac{\partial u}{\partial x}=2x \quad \frac{\partial v}{\partial y}=-1 \quad \frac{\partial u}{\partial y}=0,\ \frac{\partial v}{\partial x}=0$$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \Rightarrow 2x=-1 \quad x=-\frac{1}{2}$$

在直线 $x=-\frac{1}{2}$ 上可导，但在复平面上处处不解析.

(2) $u=2x^3$，$v=3y^3$

$$\frac{\partial u}{\partial x}=6x^2 \quad \frac{\partial v}{\partial y}=9y^2$$

$\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y} \Rightarrow$ $\sqrt{2}x\pm\sqrt{3}y=0$ 上可导，但在复平面上处处不解析.

(3) $u = xy^2$, $v = x^2y$.

$$\frac{\partial u}{\partial x} = y^2,\quad \frac{\partial v}{\partial y} = x^2\quad \frac{\partial u}{\partial y} = 2xy,\quad \frac{\partial v}{\partial x} = 2xy.$$

$$y^2 = x^2,\quad 2xy = -2xy \Rightarrow x=0,\ y=0.$$

∴ 在 $z=0$ 处可导，但在复平面上处处不解析.

(4) $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2,\quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial y} = -6xy,\quad \frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ （任意 z 在复平面上）.

在复平面上处处可导，处处解析。

2. $u = my^3 + nx^2y$, $v = x^3 + lxy^2$

$$\frac{\partial u}{\partial x} = 2nxy\quad \frac{\partial v}{\partial y} = 2lxy$$

$$\frac{\partial u}{\partial y} = 3my^2 + nx^2,\quad \frac{\partial v}{\partial x} = 3x^2 + ly^2$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow$$

$$\begin{cases} 2n = 2l \\ 3m = -l \\ n = -3 \end{cases} \Rightarrow \begin{cases} n = -3 \\ l = -3 \\ m = 1 \end{cases}$$

3.(1) 解：$z=0$ 或 $z^2=-1$　　(2) $z+1=0$ 或 $z^2+1=0$

∴ $z=0$ 或 $z=\pm i$　　　$z=-1$ 或 $z=\pm i$

4. 解：(1) $f'(z)=\frac{\partial u}{\partial x}+i\cdot\frac{\partial v}{\partial x}$

$$|f(z)|=\sqrt{u^2+v^2}$$

$$\frac{\partial|f(z)|}{\partial x}=\frac{1}{\sqrt{u^2+v^2}}\left(u\cdot\frac{\partial u}{\partial x}+v\cdot\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial|f(z)|}{\partial y}=\frac{1}{\sqrt{u^2+v^2}}\left(u\cdot\frac{\partial u}{\partial y}+v\cdot\frac{\partial v}{\partial y}\right)$$

左边 $=\frac{1}{u^2+v^2}\left[\left(u\frac{\partial u}{\partial x}+v\cdot\frac{\partial v}{\partial x}\right)^2+\left(u\frac{\partial u}{\partial y}+v\cdot\frac{\partial v}{\partial y}\right)^2\right]$

右边 $=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$

∵ $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ 代入左边，

左边 $=\left[u^2\cdot\left(\frac{\partial u}{\partial x}\right)^2+v^2\left(\frac{\partial u}{\partial x}\right)^2+2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u^2\left(-\frac{\partial v}{\partial x}\right)^2+v^2\left(\frac{\partial u}{\partial x}\right)^2+2uv\frac{\partial u}{\partial x}\left(-\frac{\partial v}{\partial x}\right)\right]\cdot\frac{1}{u^2+v^2}$

$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=$ 右边

∴ 得证

(2) $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=\frac{\partial(-u)}{\partial y}$，$\frac{\partial(-u)}{\partial x}=-\frac{\partial v}{\partial y}$

∴ $-u$ 为 v 的共轭调和函数

(此定义在第二章有介绍. 调和函数).

(3). $\frac{\partial^2 |f(z)|^2}{\partial x^2} = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)^2 + 2u\cdot\frac{\partial^2 u}{\partial x^2} + 2v\cdot\frac{\partial^2 v}{\partial x^2}$

$\frac{\partial^2 |f(z)|^2}{\partial y^2} = 2\left(\frac{\partial u}{\partial y}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2u\cdot\frac{\partial^2 u}{\partial y^2} + 2v\cdot\frac{\partial^2 v}{\partial y^2}$

由于在区域内解析, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}$ $\quad$ $\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial x\partial y}$

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ 同理可得 $\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}=0$.

∴ 左边 $=4(u_x^2+v_x^2)=4|f'(z)|^2=$ 右边

5. 设 $f(z)=u+iv$

$\overline{\overline{if(z)}} = iu-v \quad -v+iu$

∵ $f(z)$ 解析

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$\frac{\partial (-v)}{\partial x}=-\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}$, $\frac{\partial (-v)}{\partial y}=-\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}$

∴ 可证 $\overline{\overline{if(z)}}$ 在D内也解析.

6. (1). $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=-(6xy+3x^2-3y^2)$

$v=\int\frac{\partial v}{\partial x}dx=\int(-6xy+3x^2-3y^2)dx=(-3x^2y+x^3-3xy^2+\varphi(y))$

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\Rightarrow \quad 3x^2+6xy-3y^2=-3x^2+6xy-\varphi'(y)$

$\varphi'(y)=3y^2$

$\varphi(y)=\int\varphi'(y)dy=y^3+C$

∴ $v=3x^2y-x^3+3xy^2-y^3-C$

(2). $\frac{\partial u}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}$ $\frac{\partial u}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$= \frac{-2xy}{(x^2+y^2)^2} - i\frac{x^2-y^2}{(x^2+y^2)^2}$

$= -\frac{1}{z^2}$

$f(z) = \int f'(z)\,dz = -\int \frac{1}{z^2}dz = \frac{1}{z} + C$

$\because f(2) = \frac{1}{2} + C = 0 \Rightarrow C = -\frac{1}{2}$

$f(z) = \frac{1}{z} - \frac{1}{2}$

(3) $\frac{\partial u}{\partial x} = 2y$, $\frac{\partial u}{\partial y} = 2(x-1)$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $v = \int \frac{\partial u}{\partial x}dy = \int 2y\,dy = y^2 + C(x)$

$\frac{\partial v}{\partial x} = C'(x) = -2(x-1)$

$C(x) = \int C'(x)dx = \int -2(x-1)dx = -x^2+2x + C$

$\therefore v = y^2 + 2x - x^2 + C$

$\therefore f(z) = 2(x-1)y + i(y^2+2x-x^2+C)$

$\because f(0) = -i \Rightarrow C = -1$

$\therefore f(z) = 2(x-1)y + i(y^2+2x-x^2-1)$

(4) $\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times y \cdot \frac{-1}{x^2} = \frac{-y}{x^2+y^2} = \frac{\partial v}{\partial y}$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\frac{x}{x^2+y^2}$

$v = \int \frac{\partial v}{\partial y} dy = \int \left(\frac{-y}{x^2+y^2}\right) dy = -\frac{1}{2}\ln(x^2+y^2) + c(x).$

$\frac{\partial v}{\partial x} = \frac{-x}{x^2+y^2} + c'(x) = -\frac{x}{x^2+y^2}$

$\therefore c'(x) = 0$

$\therefore v = -\frac{1}{2}\ln(x^2+y^2) + c.$

$\therefore f(z) = \arctan\frac{y}{x} + i\cdot\left(-\frac{1}{2}\ln(x^2+y^2) + c\right).$

(5) $\frac{\partial u}{\partial x} = 2x + y = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = x - 2y = -\frac{\partial v}{\partial x}$

$v = \int \frac{\partial v}{\partial y} dy = \int (2x+y)\, dy = 2xy + \frac{1}{2}y^2 + c(x)$

$\frac{\partial v}{\partial x} = 2y + c'(x) = 2y - x \Rightarrow$

$c'(x) = -x \Rightarrow c(x) = \int c'(x)dx = -\frac{1}{2}x^2 + c$

$\therefore v = 2xy + \frac{1}{2}y^2 - \frac{1}{2}x^2 + c.$

$\therefore f(z) = x^2 + xy - y^2 + i\left(2xy + \frac{1}{2}y^2 - \frac{1}{2}x^2 + c\right)$

7. (1) $u = c_1(ax+by) + c_2$

(2) $u = c_1 \arctan\frac{y}{x} + c_2.$

8 证明：设 u, v 为一对共轭调和函数

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$\frac{\partial^2 (uv)}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + 2\cdot\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} \quad ①$$

$$\frac{\partial^2 (uv)}{\partial y^2} = \frac{\partial^2 u}{\partial y^2} + 2\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial y^2} \quad ②$$

$$\frac{\partial^2 (uv)}{\partial x^2} + \frac{\partial^2 (uv)}{\partial y^2} = ① + ② = 0$$

∴ 得证一对共轭调和函数的乘积仍为调和函数

9. (1) $|e^{i-2x}| = |e^{i}\cdot e^{-2x}| = |(\cos 1 + i\cdot\sin 1)\cdot e^{-2x}| = e^{-2x}$

(2) $|e^{z^2}| = |e^{x^2-y^2+2xyi}| = e^{x^2-y^2}$

(3) $\mathrm{Re}(e^{\frac{1}{z}}) = \mathrm{Re}\left(e^{\frac{x-iy}{x^2+y^2}}\right) = \mathrm{Re}\left(e^{\frac{x}{x^2+y^2}}\cdot e^{i\left(-\frac{y}{x^2+y^2}\right)}\right)$

$= \mathrm{Re}\left(e^{\frac{x}{x^2+y^2}}\cdot\left(\cos\left(\frac{-y}{x^2+y^2}\right) + i\cdot\sin\left(\frac{-y}{x^2+y^2}\right)\right)\right)$

$= e^{\frac{x}{x^2+y^2}}\cdot\cos\left(\frac{-y}{x^2+y^2}\right)$

10.

(1) $\overline{e^{z}} = \overline{e^{x}(\cos y + i\cdot\sin y)} = e^{x}\cdot(\cos y - i\cdot\sin y)$

$= e^{x}[\cos(-y) + \sin(-y)] = e^{x}\cdot e^{-iy} = e^{x-iy} = e^{\bar z}$ 正确

(2) $\overline{\cos z} = \cos\bar z$ 正确

$\overline{\cos z} = \cos x\,\mathrm{ch}\,y + i\cdot\sin x\cdot\mathrm{sh}\,y$

$\cos\bar z = \cos\overline{(x+iy)} = \cos x\cdot\mathrm{ch}\,y + i\sin x\cdot\mathrm{sh}\,y$

$\therefore \overline{\cos z} = \cos\bar z$

11. (1) $\sin z = 0$

解: $\dfrac{e^{iz} - e^{-iz}}{2i} = 0$

$e^{2iz} = 1$

$\cos 2z = 1$.

$z = k\pi \quad k \in Z$.

(2) $e^{z} = 1+\sqrt{3}i$

解:

$e^{(x+iy)} = 1+\sqrt{3}i$

$e^{x}(\cos y + i\cdot\sin y) = 1+\sqrt{3}i$

$e^{x}(\cos y + i\cdot\sin y) = 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$

$e^{x} = 2, \quad y = \frac{\pi}{3} \quad x = \ln 2$

$\therefore z = \ln 2 + i\cdot\frac{\pi}{3}$

(3). $1+e^{z} = 0$

解: $e^{z} = -1$.

$e^{(x+iy)} = 1\cdot(-1)$

$e^{x}\cdot e^{iy} = 1\cdot(\cos\pi + i\cdot\sin\pi)$

$e^{x} = 1, \quad y = \pi \quad \Rightarrow \quad x = 0, \ y = \pi$

$z = i\pi$

12. (1) $\cos(1+i)$

解 $\cos(1+i) = \dfrac{e^{i(1+i)} + e^{-i(1+i)}}{2} = \dfrac{e^{i-1} + e^{1-i}}{2}$

$= \operatorname{ch}(i-1)$

(2) $\sin(3+2i)$

解 $\sin(3+2i) = \dfrac{e^{i(3+2i)} - e^{-i(3+2i)}}{2i}$

$= \dfrac{e^{2-3i} - e^{-(2-3i)}}{2} i \quad = i\cdot\operatorname{sh}(2-3i)$

$= \operatorname{ch}2\cdot\sin 3 + i\cdot\operatorname{sh}2\cdot\cos 3$.

(3) $\tan(2-i)$

$$= \frac{\sin(2-i)}{\cos(2-i)} = \frac{\frac{e^{i(2-i)}-e^{-i(2-i)}}{2i}}{\frac{e^{i(2-i)}+e^{-i(2-i)}}{2}}$$

$$= \frac{\sin 4 - i\cdot \operatorname{sh} 2}{2(\operatorname{sh}^2 1 + \cos^2 2)}$$

(4) i^{i+1}

解 $i^{i+1} = e^{(i+1)\operatorname{Ln} i} = e^{(i+1)\left[(\frac{\pi}{2}+2k\pi)i\right]}$

$$= i \cdot e^{-(\frac{\pi}{2}+2k\pi)}$$

(5). 2^i

解 $2^i = e^{i\operatorname{Ln}2} = e^{i\cdot(2k\pi i + \ln 2)}$

$$= e^{-2k\pi}\left[\cos(\ln 2) + i\cdot\sin(\ln 2)\right]$$

(6) $\operatorname{Ln}(-3+4i)$

解: $\operatorname{Ln}(-3+4i) = \ln 5 + i\left[(2k+1)\pi - \arctan\frac{4}{3}\right]$

$(k=0, \pm 1, \cdots)$.

13. (1) 证明: 令 $z_1 = r_1\cdot e^{i\theta_1}$, $z_2 = r_2\cdot e^{i\theta_2}$

$$\operatorname{Ln}(z_1\cdot z_2) = \operatorname{Ln}(r_1\cdot r_2\cdot e^{i(\theta_1+\theta_2)})$$

$$= \ln(r_1\cdot r_2) + i(\theta_1+\theta_2+2k\pi)$$

$$\operatorname{Ln} z_1 + \operatorname{Ln} z_2 = \ln r_1 + i(\theta_1+2k\pi) + \ln r_2 + i(\theta_2+2k\pi)$$

$$= \ln(r_1\cdot r_2) + i(\theta_1+\theta_2+2k\pi) = \operatorname{Ln}(z_1\cdot z_2)$$

∴ 得证.

(2) $\mathrm{Ln}\left(\frac{z_1}{z_2}\right) = \mathrm{Ln}\left(\frac{r_1}{r_2}\cdot e^{i(\theta_1-\theta_2)}\right)$　　$k\in Z$

$= \ln\left(\frac{r_1}{r_2}\right) + i(\theta_1-\theta_2+2k\pi)$

$\mathrm{Ln}z_1 - \mathrm{Ln}z_2 = \ln r_1 + i(\theta_1+2k_1\pi) - (\ln r_2 - i(\theta_2+2k_2\pi))$

$= \ln\frac{r_1}{r_2} + i(\theta_1-\theta_2+2k_1\pi-2k_2\pi) = \mathrm{Ln}\left(\frac{z_1}{z_2}\right)$

∴ 得证.

14. (1) 由13题(1)可知左边的k只能取1，2，3，4

则右边式子中的k只可以取2，4，6，8，

即右左两边z的取值范围不同，所以(1)式不正确.

(2) 理由同(1)，z的取值范围不同，∴(2)不正确

15. (1) 证明： $\mathrm{sh}^2 z + \mathrm{ch}^2 z = \left(\frac{e^z-e^{-z}}{2}\right)^2 + \left(\frac{e^z+e^{-z}}{2}\right)^2$

$= \frac{2(e^{2z}+e^{-2z})}{4} = \mathrm{ch}2z$　∴得证

(2) $\mathrm{sh}z_1\mathrm{ch}z_2 + \mathrm{ch}z_1\cdot\mathrm{sh}z_2 = \frac{e^{z_1}-e^{-z_1}}{2}\cdot\frac{e^{z_2}+e^{-z_2}}{2} +$

$\frac{e^{z_1}+e^{-z_1}}{2}\cdot\frac{e^{z_2}-e^{-z_2}}{2}$

$= \frac{2(e^{z_1+z_2}-e^{-(z_1+z_2)})}{4} = \mathrm{sh}(z_1+z_2)$

∴ 得证

16. 证明: $z-1 = r\cos\theta - 1 + i \cdot r\sin\theta$

$$\mathrm{Re}\,\mathrm{Ln}(z-1) = \ln|z-1| = \ln\sqrt{(r\cos\theta-1)^2 + (r\sin\theta)^2}$$

$$= \frac{1}{2}\ln(r^2 - 2r\cos\theta + 1)$$

17 (1) $\mathrm{sh}z = 0$

解: $$\frac{e^z - e^{-z}}{2} = 0$$

$$e^{2z} = 1$$

$$2z = \mathrm{Ln}1$$

$$z = \frac{1}{2} \cdot 2k\pi i = k\pi i$$

$(k = 0, \pm 1, \cdots)$

(2) $\mathrm{sh}z = i$

$$\mathrm{sh}z = -i \cdot \sin iz = i$$

$$\sin iz = -1$$

$$iz = -\frac{\pi}{2} + 2k\pi$$

$$z = \left(\frac{\pi}{2} + 2k\pi\right) i$$

$(k = 0, \pm 1, \cdots)$

18 证明 $\mathrm{sh}w = z$

$$\frac{e^w - e^{-w}}{2} = z$$

$$e^{2w} - 2z \cdot e^w - 1 = 0$$

$$e^w = \frac{2z + \sqrt{4z^2+4}}{2} = z + \sqrt{z^2+1}$$

$$\therefore w = \mathrm{Ln}(z + \sqrt{z^2+1})$$

19 (1) $f(z) = u + iv$ 在区域 D 内解析,

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\overline{f(z)} = u - iv$$

f(z)在区域内解析.

$$\frac{\partial u}{\partial x}=\frac{\partial(-v)}{\partial y}=-\frac{\partial v}{\partial y}\ ,\ \frac{\partial u}{\partial y}=-\frac{\partial(-v)}{\partial x}=\frac{\partial v}{\partial x}$$

∴ $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ 均为零.

∴ f(z) 为常数.

(2) $|f(z)|=\sqrt{u^2+v^2}$

∵ |f(z)|在 D内是一个常数

∴ $\frac{\partial|f(z)|}{\partial x}=0$, $\frac{\partial|f(z)|}{\partial y}=0$. $\Rightarrow\begin{cases}u\cdot\frac{\partial u}{\partial x}+v\cdot\frac{\partial v}{\partial x}=0\\u\cdot\frac{\partial u}{\partial y}+v\cdot\frac{\partial v}{\partial y}=0.\end{cases}$

与 $\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}$ 联立 $\Rightarrow$ $\frac{\partial v}{\partial y}=0$, $\frac{\partial u}{\partial y}=0$ $\frac{\partial u}{\partial x}=0$, $\frac{\partial v}{\partial x}=0$.

∴ $f(z)=u+iv$ 为常数.

(3) 先证明 $u>0$, $v>0$ 的情况

$\arg f(x)=\arctan\frac{y}{x}$

∵ $\arg f(x)$在D内为常数

∴ $\begin{cases}\frac{\partial\arg f(x)}{\partial x}=0\\\frac{\partial\arg f(x)}{\partial y}=0\end{cases}$ $\Rightarrow\begin{cases}\frac{\partial v}{\partial x}\cdot u=\frac{\partial u}{\partial x}\cdot v\\\frac{\partial v}{\partial y}\ u=\frac{\partial u}{\partial y}\cdot v\end{cases}$

与 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ 联立,可得

$\frac{\partial u}{\partial x}=0$ $\frac{\partial v}{\partial y}=0$ $\frac{\partial v}{\partial x}=0$, $\frac{\partial u}{\partial y}=0$.

∴ f(z) 为常数。

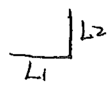
第三章

1\. (1) $\int_0^{1+i} z^2\,dz$

$= \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3}+\frac{2}{3}i$

(2) $\int_L (x^2-y^2+2xyi)\,dz$

$= \int_L (x^2-y^2)dx - 2xy\,dy + i\int_L 2xy\,dx + (x^2-y^2)\,dy$



$= \int_{L_1} x^2 dx + \int_{L_2} (-2y)\,dy + i\int_{L_2} (1-y^2)\,dy$

$= \int_0^1 x^2 dx + \int_0^1 (-2y)\,dy + i\int_0^1 (1-y^2)\,dy$

$= \frac{1}{3} - 1 + \frac{2}{3}i = -\frac{2}{3} + \frac{2}{3}i$

(3) $\int_L x^2-y^2+2xyi\,dz$

$= i\int_{L_1} (-y^2)\,dy + \int_{L_2} (x^2-1)\,dx + i\int_{L_2} 2x\,dx$

$= -i\int_0^1 -y^2\,dy + \int_0^1 (x^2-1)\,dx + i\int_0^1 2x\,dx$

$= -\frac{2}{3} + \frac{2}{3}i$

2\. $\int_L y\,dz = \int_L y\,dx + i\int_L y\,dy = i\int_0^1 y\,dy = \frac{i}{2}$

3\. $\oint_C \frac{\bar{z}}{|z|^2}dz = \oint_C \frac{\bar{z}}{z\cdot\bar{z}}dz = \oint_C \frac{1}{z}dz$

(1) $|z|=1$ $\qquad \oint_C \frac{1}{z}dz = 2\pi i$

(2) $|z|=2$ $\qquad \oint_C \frac{1}{z}dz = 2\pi i$

4. (1) $\oint_C \frac{dz}{z^2+2z+2}=0$

$z^2+2z+2=0$,

$(z+1)^2=-1=i^2$

$z+1=i$, $z+1=-i$ $\Rightarrow$ $z=i-1$, $z=-i-1$

奇点在 $|z|=1$ 的圆周外部，所以在圆周内部处处解析.

$\therefore \oint_C \frac{dz}{z^2+2z+2}=0$.

(2) $\oint_C \frac{e^z\,dz}{z^2+5z+6}$

$z^2+5z+6=0 \Rightarrow z=-2,\ z=-3$

奇点均在 $|z|=1$ 的圆周外部，圆周内部处处解析.

$\therefore \oint_C \frac{e^z}{z^2+5z+6}\,dz=0$

(3) $\oint_C z^2\cdot\cos z\,dz$

由于 $z^2\cdot\cos z$ 在复平面内处处解析，$\therefore \oint_C z^2\cdot\cos z\,dz=0$.

(4) $\oint_C \frac{1}{2z-1}\,dz$

$=\frac{1}{2}\oint_C \frac{1}{z-\frac{1}{2}}\,dz$

由于奇点 $z=\frac{1}{2}$ 在圆周 $|z|=1$ 的内部，所以 $\oint_C \frac{1}{z-\frac{1}{2}}\,dz=2\pi i$

$\therefore \oint_C \frac{1}{2z-1}\,dz=\pi i$

5 解: $z=-2$为奇点，由于奇点在$|z|=1$的圆周外，所以$\oint_C \frac{dz}{z+2}=0$，

设$z=\cos\theta+i\cdot\sin\theta$

$$\oint_C \frac{1}{z+2}dz=\oint_C \frac{1}{(\cos\theta+2)+i\cdot\sin\theta}d(\cos\theta+i\cdot\sin\theta)$$

$$=\oint_C \frac{-\sin\theta+i\cdot\cos\theta}{(\cos\theta+2)+i\cdot\sin\theta}d\theta$$

$$=\oint_C \frac{i(1+2\cos\theta)-2\sin\theta}{5+4\cos\theta}d\theta$$

由于整体的积分为0，所以其实部,虚部的积分均为0.

∴得证.

6 (1) 解: $\oint_C \frac{dz}{z^2-a^2}=\frac{1}{2a}\oint_C\left(\frac{1}{z-a}-\frac{1}{z+a}\right)dz$

$$=\frac{1}{2a}(2\pi i+0)=\frac{\pi i}{a}$$

(2) 令 $z^2-1=0$， $z^3-1=0 \Rightarrow z=\pm1$

奇点均在 $|z|=r<1$ 的范围之外，∴在积分式内处处解析.

$$\therefore \oint_C \frac{dz}{(z^2-1)(z^3-1)}=0$$

(3) 解: $\oint_C \frac{dz}{(z^2+1)(z^2+4)}=\frac{1}{3}\oint_C\left(\frac{1}{z^2+1}-\frac{1}{z^2+4}\right)dz$

$$=\frac{1}{6i}\oint_C \frac{1}{z-i}-\frac{1}{z+i}dz$$

$$=\frac{1}{6i}(2\pi i-2\pi i)=0.$$

(4) $\oint_C \frac{\sin z}{z-i} dz$

$= 2\pi i \sin z \Big|_{z=i} = 2\pi i \sin i$

(5). $\oint_C \frac{1}{z^2+4} dz$ $|z-i|=1$

$= \frac{1}{4i} \oint_C \frac{1}{z-2i} - \frac{1}{z+2i} dz$

$= \frac{1}{4i} \oint_C \frac{1}{z-2i} dz$

$= \frac{1}{4i} 2\pi i = \frac{\pi}{2}$

(6). $\oint_C \frac{\tan z}{z} dz$ $C: |z|=1$

$= 2\pi i \tan z \Big|_{z=0} = 2\pi i \cdot \tan 0 = 0$

7(1)解： $= \frac{1}{3}(z+2)^3 \Big|_{-2}^{-2+i}$

$= \frac{1}{3}[(i)^3 - 0^3] = -\frac{1}{3}i$

(2) 解：$= z^2 \cdot \sin z \Big|_0^i - \int_0^i 2z \cdot \sin z dz$

$= -\sin i + 2z\cos z \Big|_0^i - 2\int_0^i \cos z dz$

$= -\sin i + 2i \cdot \cos i - 2\sin i$

$= -3\sin i + 2i\cos i$

(3) $\int_{-\pi}^{\pi} \sin^2 z \, dz$

$$= \int_{-\pi}^{\pi} \frac{1-\cos 2z}{2} dz$$

$$= \int_{-\pi}^{\pi} \frac{1}{2} dz - \frac{1}{4}\int_{-\pi}^{\pi} \cos 2z \, d(2z)$$

$$= \frac{1}{2} z \Big|_{-\pi}^{\pi} - \frac{1}{4} \sin 2z \Big|_{-\pi}^{\pi} = \pi$$

(4) $\int_0^i (z-i)\cdot e^{-z} dz$

$$= \int_0^i z\cdot e^{-z} dz - i\int_0^i e^{-z} dz$$

$$= -\int_0^i z \cdot d(e^{-z}) + i\, e^{-z}\Big|_0^i$$

$$= -\left[z\cdot e^{-z}\Big|_0^i - \int_0^i e^{-z} dz\right] + i\cdot e^{-i} - i$$

$$= 1-\cos 1 + i(\sin 1 - 1)$$

8. (1) $\oint_C \frac{\sin z}{(z-1)^2} dz \quad C: |z|=2$

$\sin z$在复平面内处处解析，

$$\oint_C \frac{\sin z}{(z-1)^2} dz = 2\pi i \cdot \frac{\sin' z}{1!}\Big|_{z=1} = 2\pi i \cos 1.$$

(2) $\oint_{C_1+C_2} \frac{\cos z}{z^3} dz = \oint_{C_1} \frac{\cos z}{z^3} dz - \oint_{-C_2} \frac{\cos z}{z^3} dz$

$$= \frac{2\pi i}{2!} (\cos z)''\Big|_{z=0} - \frac{2\pi i}{2!} \cdot (\cos z)''\Big|_{z=0}$$

$$= 0.$$

(3) $\oint_C \frac{e^z}{(z-i)^3}dz \qquad C: |z|=2.$

$$\oint_C \frac{e^z}{(z-i)^3}dz = 2\pi i\,\frac{e^i}{2!} = i\pi\cdot e^i$$

(4). $\oint_C \frac{e^z}{(z-1)^2\cdot(z+1)^2}dz \qquad C: |z|=2,$

$|z|<2$，∴ $z=1$ 和 $z=-1$ 均为奇点

$$=\left(\frac{\left(\frac{e^z}{(z+1)^2}\right)'\Big|_{z=1}}{1!}+\frac{\left(\frac{e^z}{(z-1)^2}\right)'\Big|_{z=-1}}{1!}\right)\cdot 2\pi i$$

$$= \pi i\cdot e^{-1}$$

(5) $\oint_C \frac{1}{(z^2+4)^2}dz = \oint_C \frac{dz}{(z+2i)^2(z-2i)^2}$

$$= \oint_{C_1} \frac{\frac{1}{(z+2i)^2}}{(z-2i)^2}dz + \oint_{C_2} \frac{\frac{1}{(z-2i)^2}}{(z+2i)^2}dz$$

$$= 2\pi i\cdot[-2(2i+2i)^{-3}] + 2\pi i[-2(-2i-2i)^{-3}] = 0$$

(6) $\oint_C \frac{dz}{(z^2+9)^2} \qquad C: |z-2i|=2$

$z^2+9=0$，$\Rightarrow$ $z=\pm 3i$，$z=3i$ 在 C 内

$$= \oint_C \frac{1}{(z+3i)^2(z-3i)^2}dz$$

$$= \oint_C \frac{\frac{1}{(z+3i)^2}}{(z-3i)^2}dz = \frac{\pi}{54}$$

9. 证明：C_1 以0为圆心，r为半径的圆周，$z=r\cdot e^{i\theta}$ $\theta[0,2\pi]$

令 C_1 在C的内部，

$$\therefore \oint_C \frac{1}{z^2}\,dz=\oint_{C_1}\frac{1}{z^2}\,dz$$

$$=\int_0^{2\pi}\frac{ir\cdot e^{i\theta}}{r^2\cdot e^{2i\theta}}\,d\theta=\int_0^{2\pi}\frac{i}{r\cdot e^{i\theta}}\,d\theta=0$$

∴ 得证

10. 证明：$\oint_C \frac{f(z)}{(z-z_0)^2}\,dz=2\pi i\,f'(z_0)=0$

$\therefore f'(z)=0$ ∴ $f(z)$ 在以闭曲线C为边界所围成区域内为常数

11. 证明：由柯西积分公式得

$$\frac{1}{2\pi i}\oint_C\frac{f(z)}{z-z_0}\,dz=f(z_0)$$

∵ z_0 的取值是任意的

$$\Rightarrow \frac{c}{2\pi i}\oint_C\frac{1}{z-z_0}\,dz=\frac{c}{2\pi i}\cdot 2\pi i=c=f(z_0)$$

∴ 得证在D上为常数

12. 证明：∵ $f(z)=g(z)$ 在 C 上所有的点处成立

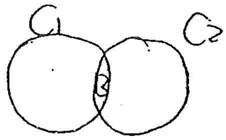
∴ $\oint_C [f(z)-g(z)]dz=0$

∵ C 在 D 的内部，C 内处处解析，

由复合闭路定理，得 在 C 内部的所有点均有

$\oint_\Gamma f(z)dz=0$ 成立， 得证

13. 证明：



由复合闭路定理，得

$\oint_{C_1} f(z)dz = \oint_B f(z)dz$

$\oint_{C_2} f(z)dz = \oint_B f(z)dz$

∴ $\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$

14. 证明：由于在区域 D C 曲线及其内部处处解析，

∴ $\oint_C \frac{f'(z)}{z-z_0}dz = 2\pi i f'(z_0)$

$\oint_C \frac{f(z)}{(z-z_0)^2}dz = f(z_0)\cdot 2\pi i$

∴ $\oint_C \frac{f'(z)}{z-z_0}dz = 2\pi i f'(z_0) = \oint_C \frac{f(z)}{(z-z_0)^2}dz$

∴ 得证

15 证明：设 z_0 在 $|z|\le r$ 的内部，C 任意取，

$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right|$$

$$= \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right|$$

$$\le \frac{n!}{2\pi}\cdot\frac{M}{|(z-z_0)|^{n+1}}\cdot 2\pi\cdot|(z-z_0)|$$

$$\le \frac{n!\cdot M}{(r-|z_0|)^{n}}$$

$$\therefore |f^{(n)}(z)| \le \frac{n!\cdot M}{(r-|z|)^{n}}$$，得证

18. 证明：假设 $|f(z_0)|$ 是 $|f(z)|$ 在 D 的最小值，即 $|f(z_0)|=m$。

而 $f(z)$ 在 D 内解析且不为常数，

由保域定理知 $G=f(D)$ 为 w 平面上的区域。

因 $f(z_0)=w_0\in G$，则 $\exists\ (w_0,\varepsilon)\subset G$，又 $f(z_0)=w_0\ne 0$，

因此 $\exists\, w_1\in(w_0,\varepsilon)$ 满足 $|w_1|<|w_0|$，故 $\exists\, z_1\in D$，

使得 $f(z_1)=w_1$ 且 $|f(z_1)|<|f(z_0)|=m$，

这显然与 m 为 $|f(z)|$ 在 D 内的最小值矛盾，

所以 $|f(z_0)|$ 不可能是 $|f(z)|$ 在 D 内的最小值。

第四章

八、(1) 解：令 $a_n=\frac{1}{n}$，$b_n=\frac{1}{2^n}$

∵ $\lim\limits_{n\to\infty} a_n=\lim\limits_{n\to\infty}\frac{1}{n}=0$，$\lim\limits_{n\to\infty} b_n=\lim\limits_{n\to\infty}\frac{1}{2^n}=0$

∴ 复数列 $z_n=\frac{1}{n}+\frac{i}{2^n}$ 收敛，$\lim\limits_{n\to\infty} z_n=0$

(2) 解：∵ $z_n=e^{-\frac{n\pi i}{2}}=\cos\left(-\frac{n\pi}{2}\right)+i\sin\left(-\frac{n\pi}{2}\right)$

$=\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}$

∵ $\lim\limits_{n\to\infty}\cos\frac{n\pi}{2}$、$\lim\limits_{n\to\infty}\sin\frac{n\pi}{2}$ 均不存在

∴ 复数列 $z_n=e^{-\frac{n\pi i}{2}}$ 发散.

(3) 解：$z_n=(1+\sqrt{3}i)^{-n}=\left[2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^{-n}$

$=2^{-n}\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right)$

∵ $a_n=\frac{1}{2^n}\cos\frac{n\pi}{3}$，$b_n=\frac{-1}{2^n}\sin\frac{n\pi}{3}$

∴ $\lim\limits_{n\to\infty} a_n=0$，$\lim\limits_{n\to\infty} b_n=0$

∴ 复数列 z_n 收敛，$\lim\limits_{n\to\infty} z_n=0$

(4) 解：$z_n=\left(1+\frac{1}{n}\right)e^{i\frac{\pi}{n}}=\left(1+\frac{1}{n}\right)\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)$

∵ $a_n=\left(1+\frac{1}{n}\right)\cos\frac{\pi}{n}$，$b_n=\left(1+\frac{1}{n}\right)\sin\frac{\pi}{n}$

$\lim\limits_{n\to\infty} a_n=1$，$\lim\limits_{n\to\infty} b_n=0$

∴ 复数列 z_n 收敛，$\lim\limits_{n\to\infty} z_n=1$

2. (1) 解：∵ $\sum_{n=0}^{\infty}\left|\frac{(3i)^n}{n!}\right| = \sum_{n=0}^{\infty}\frac{3^n}{n!}$ 收敛．

$\left(\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n} = \lim_{n\to\infty}\frac{3}{n+1} = 0 < 1\right)$

∴ $\sum_{n=0}^{\infty}\frac{(3i)^n}{n!}$ 绝对收敛

(2) 解：∵ $\sum_{n=2}^{\infty}\frac{i^n}{\ln n} = \sum_{n=2}^{\infty}\frac{(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})^n}{\ln n}$

$= \sum_{n=2}^{\infty}\frac{\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}}{\ln n}$

而 $\sum a_n = \sum_{n=2}^{\infty}\frac{\cos\frac{n\pi}{2}}{\ln n}$ 与 $\sum_{n=1}^{\infty} b_n = \sum_{n=2}^{\infty}\frac{\sin\frac{n\pi}{2}}{\ln n}$ 收敛，故原级数收敛

又∵ $\sum_{n=2}^{\infty}\left|\frac{i^n}{\ln n}\right| = \sum_{n=2}^{\infty}\frac{1}{\ln n}$ 发散，所以原级数条件收敛．

(3) 解：$\sum_{n=0}^{\infty}\frac{\sin in}{2^n} = \sum_{n=0}^{\infty}\frac{e^{i\cdot in}-e^{-i\cdot in}}{2i\cdot 2^n} = \sum_{n=0}^{\infty}\frac{(e^n-e^{-n})i}{2^{n+1}}$

又∵ $\lim_{n\to\infty}\frac{(e^n-e^{-n})}{2^{n+1}} \neq 0$，故原级数发散

(4) 解：∵ $\sum_{n=0}^{\infty}\left|\frac{(-1)^n i^n}{2^n}\right| = \sum_{n=0}^{\infty}\frac{1}{2^n}$ 收敛，

故原级数绝对收敛．

3. 证明：令 $z_n = a_n + ib_n$

∵ 复数列 $z_1, z_2, \cdots, z_n, \cdots$ 全部位于半平面 $Re(z) \geq 0$

∴ $a_n \geq 0$

∵ $\sum_{n=1}^{\infty} z_n$ 收敛 ∴ $\sum_{n=1}^{\infty} a_n$ 和 $\sum_{n=1}^{\infty} b_n$ 均收敛．

又∵ $\sum_{n=1}^{\infty} z_n^2$ 收敛．∴ $\sum_{n=1}^{\infty} z_n^2 = \sum_{n=1}^{\infty}(a_n^2-b_n^2+2a_nb_n i)$

即 $\sum_{n=1}^{\infty}(a_n^2-b_n^2)$ 收敛，$\sum_{n=1}^{\infty} 2a_nb_n$ 收敛．

$\because \sum\limits_{n=1}^{\infty} a_n^2$、$\sum\limits_{n=1}^{\infty} b_n^2$ 均收敛

而 $\sum\limits_{n=1}^{\infty} |z_n|^2 = \sum\limits_{n=1}^{\infty}(a_n^2+b_n^2)$ 收敛

结论得证

4. (1) 解：$\because \rho = \lim\limits_{n\to\infty}\left|\dfrac{C_{n+1}}{C_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(1+2i)^{n+1}}{(1+2i)^n}\right| = \lim\limits_{n\to\infty}|1+2i| = \sqrt{5}$

$\therefore R = \dfrac{\sqrt{5}}{5}$

(2) 解：$\because \rho = \lim\limits_{n\to\infty}\left|\dfrac{C_{n+1}}{C_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{e^{\frac{i\pi}{n+1}}}{e^{\frac{i\pi}{n}}}\right| = \lim\limits_{n\to\infty}\left|e^{\frac{-i\pi}{n(n+1)}}\right|$

$= \lim\limits_{n\to\infty}\left|\cos\dfrac{\pi}{n(n+1)} - i\sin\dfrac{\pi}{n(n+1)}\right| = 1$

$\therefore R = 1$

(3) 解：$\because \cos(in) = \dfrac{e^{i\cdot in}+e^{-i\cdot in}}{2} = \dfrac{e^n+e^{-n}}{2}$

$\rho = \lim\limits_{n\to\infty}\left|\dfrac{C_{n+1}}{C_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{e^{n+1}+e^{-(n+1)}}{e^n+e^{-n}}\right| = e$

$\therefore R = \dfrac{1}{e}$

(4) 解：$\rho = \lim\limits_{n\to\infty}\left|\dfrac{C_{n+1}}{C_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)^p}{n^p}\right| = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^p = 1$

$\therefore R = 1$

(5) 解：$\rho = \lim\limits_{n\to\infty}\left|\dfrac{C_{n+1}}{C_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)^{n+1}}{[(n+1)!]^2}\Big/\dfrac{n^n}{(n!)^2}\right| = \lim\limits_{n\to\infty}\left[\dfrac{1}{n+1}\left(1+\dfrac{1}{n}\right)^n\right]$

$= 0$

$\therefore R = \infty$

(6) $\because \rho=\lim\limits_{n\to\infty}\sqrt[n]{|C_n|}=\lim\limits_{n\to\infty}\sqrt[n]{\left|\frac{1}{(\ln in)^n}\right|}=\lim\limits_{n\to\infty}\left|\frac{1}{\ln in}\right|=0$

$\therefore R=\frac{1}{\rho}=\infty$

5、

(1) 解：$\because \rho=\lim\limits_{n\to\infty}\left|\frac{n+2}{n+1}\right|=1 \quad \therefore R=\frac{1}{\rho}=1$

$S_n=\sum\limits_{n=0}^{\infty}[(z-3)^{n+2}]'-\sum\limits_{n=0}^{\infty}(z-3)^{n+1}$

$=\left[\frac{(z-3)^2}{1-(z-3)}\right]'-\frac{z-3}{1-(z-3)}=\frac{z-3}{(4-z)^2}$

$\because$ 在 $|z-3|=1$ 上，即 $\sum\limits_{n=0}^{\infty}(n+1)$ 不收敛 发散

$\therefore$ 收敛圆为 $|z-3|<1$

(2) 解：$\because \rho=\lim\limits_{n\to\infty}\left|\frac{(n+1)^{\ln(n+1)}}{n^{\ln n}}\right|=\lim\limits_{n\to\infty}\left|\frac{e^{[\ln(n+1)]^2}}{e^{[\ln n]^2}}\right|$

$=\lim\limits_{n\to\infty}e^{[\ln(n+1)+\ln n][\ln(n+1)-\ln n]}=1$

$\because$ 在 $|z-i|=1$ 时上，$\sum\limits_{n=1}^{\infty}n^{\ln n}$ 发散

$\therefore$ 收敛圆为 $|z-i|<1$

(3) 解：$\because \rho=\lim\limits_{n\to\infty}\left|\frac{(n+1)^2}{e^{n+1}}\Big/\frac{n^2}{e^n}\right|=\lim\limits_{n\to\infty}\frac{(1+\frac{1}{n})^2}{e}=\frac{1}{e}$

$\therefore R=e$

在圆周 $|z-1|=e$ 上，$\sum\limits_{n=1}^{\infty}\frac{n^2}{e^n}\cdot e^n=\sum\limits_{n=1}^{\infty}n^2$ 不收敛

$\therefore$ 收敛圆为 $|z-1|<e$.

(4)解：$\sum_{n=1}^{\infty}(n+a^n)(z+i)^n=\sum_{n=1}^{\infty}n(z+i)^n+\sum_{n=1}^{\infty}a^n(z+i)^n$

∵ $\sum_{n=1}^{\infty}n(z+i)^n$，$\rho_1=\lim_{n\to\infty}\left|\frac{n+1}{n}\right|=1$，

$\sum_{n=1}^{\infty}a^n(z+i)^n$，$\rho_2=\lim_{n\to\infty}\left|\frac{a^{n+1}}{a^n}\right|=|a|$

∴当 $|a|>1$ 时，收敛半径为 $\frac{1}{|a|}$，收敛圆为 $|z+i|<\frac{1}{|a|}$

当 $|a|<1$ 时，收敛半径为 1，收敛圆为 $|z+i|<1$

6、证明：

令 $\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|=\rho$，收敛半径为 $R=\frac{1}{\rho}$

若 $R>2$，则 $\rho=\frac{1}{2}$，那么由正项级数比值判别法可知

$\sum_{n=0}^{n}2^n|C_n|$ 收敛，与已知矛盾

若 $R<2$，因为 $\sum_{n=0}^{\infty}2^nC_n$ 收敛，即 $\sum_{n=0}^{\infty}C_nz^n$ 在 $z=2$ 收敛，

那么必有 $R\geq 2$ 成立，与假设矛盾。∴ $R=2$

7. ∵ $\sum_{n=0}^{\infty}C_nz^n$ 在它的收敛圆周 z_0 处绝对收敛

∴ 即 $\sum_{n=0}^{\infty}|C_nz_0^n|$ 收敛

即收敛半径 ~~$|z|<|z_0|$~~ $R=|z_0|$

∴ 在 $z<|z_0|$ 区域内

即在收敛圆周所围的闭区域上绝对收敛

8. $R=\frac{1}{\rho}=\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

$$R'=\frac{1}{\rho'}=\lim_{n\to\infty}\left|\frac{n^{10}a_n}{a_{n+1}(n+1)^{10}}\right|$$

$$=\lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^{10}\frac{a_n}{a_{n+1}}\right|=R$$

9.(1) $\frac{1}{1+z^3}=1-z^3+z^6-z^9+\cdots=\sum\limits_{n=0}^{\infty}(-1)^n z^{3n}$

$|z^3|<1$，∴收敛半径 $R=1$.

(2) ∵ $\frac{z^2-3z-1}{(z+2)(z-1)^2}=\frac{1}{z+2}-\frac{1}{(z-1)^2}$

又∵ $\frac{1}{(z-1)^2}=\left(\frac{1}{1-z}\right)'=(1+z+z^2+\cdots+z^n)'$ $|z|<1$

$=1+2z+3z^2+\cdots nz^{n-1}$; $|z|<1$

$\frac{1}{z+2}=\frac{1}{1+\frac{z}{2}}\cdot\frac{1}{2}=\frac{1}{2}\cdot\sum\limits_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\cdot(-1)^n$ $\left|\frac{z}{2}\right|<1$

∴ $\frac{z^2-3z-1}{(z+2)(z-1)^2}=\sum\limits_{n=0}^{\infty}\left[(-1)^n\cdot\frac{1}{2^{n+1}}-(n+1)\right]z^n$, $R=1$

(3) 由于 $\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}\cdots+(-1)^n\frac{z^{2n}}{(2n)!}+\cdots$; $|z|<+\infty$

将上式中的 z 都换成 z^2

得. $\cos z^2=\sum\limits_{n=0}^{\infty}\frac{(-1)^n z^{4n}}{(2n)!}$. $R=+\infty$.

(4) $\because \operatorname{sh} z = \dfrac{e^z - e^{-z}}{2}$

又 $\because e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots + \dfrac{z^n}{n!} + \cdots, \quad |z| < +\infty$

$e^{-z} = 1 - z + \dfrac{z^2}{2!} - \dfrac{z^3}{3!} + \cdots + (-1)^n \dfrac{z^n}{n!} + \cdots, \quad |z| < +\infty$

$\therefore \operatorname{sh} z = \dfrac{2\left[z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots \dfrac{z^{2n+1}}{(2n+1)!}\right]}{2}$

$= \sum\limits_{n=0}^{\infty} \dfrac{z^{2n+1}}{(2n+1)!}, \quad R = +\infty.$

(5) $\because C_n = \dfrac{f^{(n)}(z_0)}{n!}, \quad f(0) = 1$

$\left(e^{\frac{z}{z-1}}\right)' = \dfrac{-1}{(z-1)^2} e^{\frac{z}{z-1}}, \quad f'(0) = -1$

$\left(e^{\frac{z}{z-1}}\right)'' = \dfrac{1}{(z-1)^4} e^{\frac{z}{z-1}} + \dfrac{2}{(z-1)^3} e^{\frac{z}{z-1}}, \quad f''(0) = -1$

$\left(e^{\frac{z}{z-1}}\right)''' = -1$

$\therefore e^{\frac{z}{z-1}} = 1 - z - \dfrac{z^2}{2} - \dfrac{z^3}{6} + \cdots$

(6) $\because e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots + \dfrac{z^n}{n!} + \cdots, \quad |z| < +\infty.$

$\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} \cdots + (-1)^n \dfrac{z^{2n}}{(2n)!} + \cdots, \quad |z| < +\infty.$

根据幂级数的乘法，设

$e^z \cdot \cos z = C_0 + C_1 z + C_2 z^2 + \cdots + C_n z^n + \cdots, \quad |z| < +\infty$

于是有 $e^z \cos z = 1 + z - \dfrac{1}{3} z^3 - \dfrac{1}{6} z^4 - \dfrac{1}{30} z^5 + \cdots, \quad R = +\infty$

(7) ∵函数 $\frac{e^z}{1+z}$ 距原点最近的奇点是 -1，∴它在原点处幂级数展开式的收敛半径 $R=1$.

由于 $e^z = 1+z+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}+\cdots$，$|z|<+\infty$

$\frac{1}{1+z} = 1-z+z^2+\cdots+(-1)^n z^n+\cdots$，$|z|<1$

根据幂级数乘法，设

$$\frac{e^z}{1+z} = C_0+C_1z+C_2z^2+\cdots C_nz^n, \quad |z|<1$$

得有 $\frac{e^z}{1+z} = 1+\frac{z^2}{2!}-\frac{2}{3!}z^3+\frac{9}{4!}z^4-\frac{44}{5!}z^5+\cdots \quad R=1.$

(8) $\tan z = \frac{\sin z}{\cos z}$ ∵函数 $\frac{\sin z}{\cos z}$ 距原点最近的奇点是 $\pm\frac{\pi}{2}$，∴它在原点处幂级数展开式收敛半径为 $R=\frac{\pi}{2}$.

由于 $\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots+(-1)^n\frac{z^{2n+1}}{(2n+1)!}+\cdots \quad |z|<+\infty$

$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots+(-1)^n\frac{z^{2n}}{(2n)!}+\cdots \quad |z|<+\infty$

根据幂级数除法 设

$$\frac{\sin z}{\cos z} = C_0+C_1z+C_2z^2+\cdots+C_nz^n. \quad |z|<\frac{\pi}{2}$$

$$\therefore \tan z = z+\frac{1}{3}z^3+\frac{2}{15}z^5+\cdots, \quad R=\frac{\pi}{2}$$

(9) ∵ $C_n = \frac{f^{(n)}(z_0)}{n!}$，$f(0)=1$，$C_0=1$

$\left[\frac{1}{(1-z)^k}\right]' = \frac{k}{(1-z)^{k+1}}$，$f'(0)=k$ ∴ $C_1=k$

同理 $C_2=\frac{k(k+1)}{2!}$ $\quad C_3=\frac{k(k+1)(k+2)}{3!}$

$$\therefore \frac{1}{(1-z)^k} = 1+kz+\frac{k(k+1)}{2!}+\frac{k(k+1)(k+2)}{3!}+\cdots, \quad R=1.$$

(10) $\because f(z)=\sin\frac{1}{1-z}$ 距原点最近的奇点是1. $\therefore R=1$.

$\because C_n=\frac{f^{(n)}(z_0)}{n!}$. $f(0)=\sin 1$. $\therefore C_0=\sin 1$

$C_1=\left(\sin\frac{1}{1-z}\right)'\Big|_{z=0}=\cos 1$. 同理. $C_2=\cos 1-\frac{1}{2}\sin 1$

$C_3=\frac{5}{6}\cos 1-\sin 1$

$\therefore \sin\frac{1}{1-z}=\sin 1+\cos 1\cdot z+(\cos 1-\frac{1}{2}\sin 1)z^2+(\frac{5}{6}\cos 1-\sin 1)z^3+\cdots$

10. (1) 解: 由 $\frac{1}{z}=\frac{1}{1+(z-1)}$.

当 $|z-1|<1$ 时, 有

$$\frac{1}{1+(z-1)}=1-(z-1)+(z-1)^2+\cdots+(-1)^n(z-1)^n+\cdots$$

$\therefore \frac{1}{z}=\sum_{n=0}^{\infty}(-1)^n(z-1)^n$, $R=1$.

(2) 解:

由 $\frac{z}{(z+1)(z+2)}=\frac{2}{z+2}-\frac{1}{z+1}$

而 $\frac{2}{z+2}=\frac{1}{2}\cdot\frac{1}{1+\frac{z-2}{4}}$, 当 $\left|\frac{z-2}{4}\right|<1$ 时, 有

$$\frac{2}{z+2}=\frac{1}{2}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{4}\right)^n;$$

$\frac{1}{z+1}=\frac{1}{3}\cdot\frac{1}{1+\frac{z-2}{3}}$, 当 $\left|\frac{z-2}{3}\right|<1$ 时,

有 $\frac{1}{z+1}=\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{3}\right)^n$

$\therefore \frac{z}{(z+1)(z+2)}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2^{2n+1}}-\frac{1}{3^{n+1}}\right)(z-2)^n$, $R=3$.

(3) 解：由于 $\frac{z-1}{z+1}=\frac{z-1}{z-1+2}=\frac{\frac{z-1}{2}}{1+\frac{z-1}{2}}$

当 $\left|\frac{z-1}{2}\right|<1$ 时，有

$$\frac{z-1}{z+1}=\frac{z-1}{2}\cdot\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2+\cdots(-1)^n\left(\frac{z-1}{2}\right)^n\right]$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^n}(z-1)^n \qquad R=2$$

(4) 解：由于 $\frac{1}{3+i-2z}=\frac{1}{1-i-2(z-1-i)}=\frac{1}{1-i}\cdot\frac{1}{1-\frac{2(z-1-i)}{1-i}}$

当 $\left|\frac{2(z-i-1)}{1-i}\right|<1$ 时，有　　即 $R=\frac{\sqrt{2}}{2}$

$$\frac{1}{3+i-2z}=\frac{1}{1-i}\cdot\sum_{n=0}^{\infty}\frac{2^n(z-1-i)^n}{(1-i)^n}$$

$$=\sum_{n=0}^{\infty}\frac{2^n}{(1-i)^{n+1}}\left[z-(1+i)\right]^n.$$

(5) 解：由于 $e^z=e\cdot e^{z-1}$

$$=e\left[1+(z-1)+\frac{(z-1)^2}{2!}+\cdots+\frac{(z-1)^n}{n!}\right]$$

$$=e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}. \quad R=+\infty$$

(6) 解：由于 $\frac{1}{1+z^2}=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)$

而 $\frac{1}{z-i}=\frac{1}{1-i+z-1}=\frac{1}{1+\frac{z-1}{1-i}}\cdot\frac{1}{1-i}$

当 $\left|\frac{z-1}{1-i}\right|<1$ 时，有

$$\frac{1}{z-i}=\frac{1}{1-i}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{1-i}\right)^n$$

同理 $\frac{1}{z+i}=\frac{1}{1+i}\cdot\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{1+i}\right)^n$

$\therefore \frac{1}{1+z^2}=\frac{1}{2i}\left[\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{(1-i)^n}-\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{(1+i)^n}\right]$

$=\frac{1}{2}-\frac{1}{2}(z-1)+\frac{1}{4}(z-1)^2-\frac{1}{8}(z-1)^4+\frac{1}{8}(z-1)^5$

$-\frac{1}{16}(z-1)^6+\cdots,\quad R=\sqrt{2}.$

(7)解：
$\arctan z=\int\frac{1}{1+z^2}dz$

$=\int(1-z^2+z^4+\cdots)dz$

$=z-\frac{z^3}{3}+\frac{z^5}{5}+\cdots=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{z^{2n+1}}{2n+1}\quad R=1.$

11、证明：如果 $f(z)$ 在圆域 $D:|z-z_0|<R$ 内解析，那么 $f(z)$ 在D内可以唯一地展开成幂级数。

$\because$ 当 $f(z)$ 在 $z_0=0$ 处展开成幂级数时

$C_n=\frac{f^{(n)}(z_0)}{n!},\quad z_0=0,\ (n=0,1,2).$

又：展开式系数都是实数

10、18)解：设 $f(z)=\sqrt{z-1}$，$f(0)=-1$，$C_0=-1$

$C_n=\frac{f^{(n)}(z_0)}{n!}\qquad C_1=\left.\frac{\frac{1}{2}(z-1)^{-\frac{1}{2}}}{1!}\right|_{z=0}=-\frac{1}{2}$

$C_n=\frac{-\frac{1}{2}\cdot(\frac{1}{2}-1)\cdot(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!},\quad n=1,2,3.$

$\therefore \sqrt{z-1}=\sum_{n=1}^{\infty}\frac{-\frac{1}{2}\cdots(\frac{1}{2}-n+1)}{n!}+-1$

12、证明：∵ $e^{z}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots+\frac{z^n}{n!}$, $|z|<1$

$$\therefore |e^{z}-1| = \left|z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots+\frac{z^n}{n!}+\cdots\right|$$

$$\leq |z|+\left|\frac{z^2}{2!}\right|+\left|\frac{z^3}{3!}\right|+\cdots+\left|\frac{z^n}{n!}\right|+\cdots$$

$$= e^{|z|}-1 = |z|\sum_{n=1}^{\infty}\frac{|z|^{n-1}}{n!}$$

$$\leq |z|\sum_{n=1}^{\infty}\frac{|z|^{n-1}}{(n-1)!}$$

$$= |z|e^{|z|}$$

又∵ $|e^{z}-1| = |z|\left|\sum_{n=1}^{\infty}\frac{z^{n-1}}{n!}\right| \geq |z|\left(1-\sum_{n=2}^{\infty}\frac{|z|^{n-1}}{n!}\right)$

$$\geq |z|\left(1-\sum_{n=2}^{\infty}\frac{1}{n!}\right)$$

$$= |z|\,(3-e) > \frac{|z|}{4}$$

而 $e^{|z|}-1 = |z|\sum_{n=1}^{\infty}\frac{|z|^{n-1}}{n!} \leq |z|\sum_{n=1}^{\infty}\frac{1}{n!}$

$$= |z|\,(e-1) < \frac{7}{4}|z|$$

$\therefore\ \frac{|z|}{4} < |e^{z}-1| < \frac{7}{4}|z|$ 得证

13、(1) ∵ $f(z)$ 有一个奇点 $z=5$，所以 $f(z)$ 在以 $z=5$ 为心的圆环域解析

∴ $f(z)=\dfrac{1}{-2+z-3}=-\dfrac{1}{2}\cdot\dfrac{1}{1-\frac{z-3}{2}}$

在 $0<|z-3|<2$ 圆环内，$\left|\dfrac{z-3}{2}\right|<1$ 成立

∴ $f(z)=-\dfrac{1}{2}\left[1+\dfrac{z-3}{2}+\left(\dfrac{z-3}{2}\right)^2+\cdots+\left(\dfrac{z-3}{2}\right)^n+\cdots\right]$

$=-\dfrac{1}{2}\sum\limits_{n=0}^{\infty}\left(\dfrac{z-3}{2}\right)^n=-\sum\limits_{n=0}^{\infty}\dfrac{1}{2^{n+1}(z-3)^n}$，$0<|z-3|<2$

$f(z)=\dfrac{1}{-4+z-1}=\dfrac{1}{z-1}\cdot\dfrac{1}{1-\frac{4}{z-1}}$

在 $4<|z-1|<+\infty$ 圆环内 $\left|\dfrac{4}{z-1}\right|<1$ 成立

∴ $f(z)=\dfrac{1}{z-1}\cdot\sum\limits_{n=0}^{\infty}\left(\dfrac{4}{z-1}\right)^n=\sum\limits_{n=0}^{\infty}\dfrac{4^n}{(z-1)^n}$，$4<|z-1|<+\infty$

(2) $\dfrac{1}{(z^2+1)(z-2)}=\dfrac{1}{5}\cdot\left(\dfrac{1}{z-2}+\dfrac{z+2}{z^2+1}\right)$

$=-\dfrac{1}{10}\cdot\left[\dfrac{1}{1-\frac{z}{2}}\right]-\dfrac{1}{5}\cdot\left[\dfrac{1}{z}+\dfrac{2}{z^2}\right]\dfrac{1}{1+\frac{1}{z^2}}$

$=-\dfrac{1}{10}\left(1+\dfrac{z}{2}+\dfrac{z^2}{2^2}+\cdots\right)-\dfrac{1}{5}\left(\dfrac{1}{z}+\dfrac{2}{z^2}-\dfrac{1}{z^3}+\cdots\right)$

$=\dfrac{1}{5}\left(\cdots+\dfrac{2}{z^4}+\dfrac{1}{z^3}-\dfrac{2}{z^3}-\dfrac{1}{z}-\dfrac{1}{2}-\dfrac{z}{4}\right.$

$\left.-\dfrac{z^2}{8}-\dfrac{z^3}{16}-\cdots\right)$

$1<|z|<2$

(3) $f(z)=\frac{1}{z^2(z-i)}$

由于 $\frac{1}{z-i}=-\frac{1}{i}\cdot\frac{1}{1-\frac{z}{i}}$，在 $0<|z|<1$ 内 $\left|\frac{z}{i}\right|<1$ 成立.

$\therefore \frac{1}{z-i}=-\frac{1}{i}\sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n=-\sum_{n=0}^{\infty}\frac{z^n}{i^{n+1}}$

$\therefore f(z)=-\sum_{n=0}^{\infty}\frac{z^{n-2}}{i^{n+1}}$

(4) $f(z)=\frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)$

当 $0<|z-i|<2$ 内，$\left|\frac{z-i}{2i}\right|<1$ 成立

$\therefore \frac{1}{z+i}=\frac{1}{z-i+2i}=\frac{1}{2i}\cdot\frac{1}{1+\frac{z-i}{2i}}=\frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n$

$\therefore f(z)=\frac{1}{2i(z-i)}-\sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^n}{(2i)^{n+2}}=\sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^{n-1}}{(2i)^{n+1}}$

当在 $2<|z-i|<+\infty$ 内 $\left|\frac{2i}{z-i}\right|<1$ 成立

$\therefore \frac{1}{z+i}=\frac{1}{z-i+2i}=\frac{1}{z-i}\cdot\frac{1}{1+\frac{2i}{z-i}}=\frac{1}{z-i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2i}{z-i}\right)^n$

$\therefore f(z)=\sum_{n=0}^{\infty}(-1)^n\frac{(2i)^n}{(z-i)^{n+2}}$

(5) $f(z)=z^2\cdot e^{\frac{1}{z}}$

由于 $e^{\frac{1}{z}}=1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\cdots+\frac{1}{n!z^n}+\cdots$

$\therefore f(z)=z^2\cdot\sum_{n=0}^{\infty}\frac{1}{n!z^n}=\frac{1}{n!z^{n-2}}$

(16) 解：在圆环 $0<|z-2|<+\infty$ 内

$$f(z)=\frac{1}{z-2}-\frac{1}{3!(z-2)^3}+\cdots+(-1)^n\frac{1}{(2n+1)!(z-2)^{2n+1}}$$
$$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!(z-2)^{2n+1}}$$

14. 答：不能

∵当 $\frac{1}{z}=k\pi+\frac{\pi}{2}$，$k\in 0,\pm1,\cdots$

或 $z=0$ 时，$\tan\frac{1}{z}$ 无定义

即 $z=0$，$z=\frac{2}{2k\pi+\pi}$，即 $z=+\frac{2}{\pi},\pm\frac{2}{3\pi},\cdots\pm\frac{2}{(2n+1)\pi}$

而 $\lim\limits_{n\to\infty}\pm\frac{2}{(2n+1)\pi}=0$.

∴ $0<|z|<R$ 内取不到 R，所以原函数不能在圆环内展开

13.(7) ∵ $f(z)=e^{\frac{1}{1-z}}$ 在 $1<|z|<+\infty$ 内解析 ∴ $f(\frac{1}{z})=e^{\frac{z}{z-1}}$

在圆环域 $|z|<1$ 内解析，而在 $|z|<1$ 内

$$f\left(\frac{1}{z}\right)=e^{\frac{z}{z-1}}=1-z-\frac{z^2}{2!}-\frac{z^3}{3!}-\cdots$$

$$\therefore f(z)=e^{\frac{1}{1-z}}=1-\frac{1}{z}-\frac{1}{2!z^2}-\frac{1}{3!z^3}-\cdots$$

15. 证明：令 C 为单位圆 $|z|=1$，在 C 上取积分变量 $z=e^{i\theta}$，则

$z+\frac{1}{z}=2\cos\theta$，$dz=ie^{i\theta}$

$$C_n=\frac{1}{2\pi i}\oint_C\frac{\sin(z+\frac{1}{z})}{z^{n+1}}dz$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin(2\cos\theta)}{\cos n\theta+i\sin n\theta}d\theta.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \sin(2\cos\theta)d\theta - \frac{i}{\pi}\int_0^{2\pi} \sin n\theta \sin(2\cos\theta)d\theta$$

命令 $t=\theta-\pi$，有

$$\int_0^{2\pi} \sin n\theta \sin(2\cos\theta)d\theta = \int_{-\pi}^{\pi} (-1)^n \sin nt \sin(-2\cos t)dt$$

$$=0, \text{证毕.}$$

16、证明：当 $|z|>k$，且 $k^2<1$，在圆环域中的罗伦级数为

$$(z-k)^{-1} = \frac{1}{z}\cdot\frac{1}{1-\frac{k}{z}}$$

$$= \frac{1}{z}\left(1+\frac{k}{z}+\frac{k^2}{z^2}+\cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{k^n}{z^{n+1}}$$

取 $z=e^{i\theta}$ 代入上式得

$$(e^{i\theta}-k)^{-1} = \frac{1}{\cos\theta + i\sin\theta - k}$$

$$= \frac{\cos\theta - k - i\sin\theta}{1-2k\cos\theta + k^2}$$

而

$$\sum_{n=0}^{\infty}\frac{k^n}{z^{n+1}} = \sum_{n=0}^{\infty} k^n e^{-(n+1)i\theta}$$

$$= \sum_{n=0}^{\infty}\left[k^n\cos(n+1)\theta - ik^n\sin(n+1)\theta\right]$$

两式实部对应实部，虚部对应虚部，证毕.